ON THE TWO-DIMENSIONAL PROBLEM OF PROPAGATION OF ELASTIC WAVES DUE TO A POINT SOURCE IN AN ANISOTROPIC MEDIUM

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The two-dimensional problem of propagation of elastic waves from a point source in the form of an instantaneous impulse in an anisotropic medium having four elastic constants was examined by Sveklo [1] using the Smirnov-Sobolev method of complex solutions [2]. Since the treatment of this problem presents some difficulties, the solution obtained was studied only for media which were restricted by certain conditions on the elastic constants. In this connection it is expedient to reexamine these solutions and on this basis to study in detail the geometric form of the wave fronts. These problems are of interest in themselves and are, moreover, necessary for the solution of a number of other problems. As in [1] we shall limit ourselves to the study of quasi-longitudinal waves and quasi-transverse waves of SV type, since the study of waves of SH type presents no difficulties.

1. The equations of motion and their solutions. The equations of motion for anisotropic medium in the two-dimensional case have the form

$$a\frac{\partial^2 u}{\partial x^2}+d\frac{\partial^2 u}{\partial y^2}+c\frac{\partial^2 v}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial t^2}\qquad\left(a=\frac{c_{11}}{\rho},\quad b=\frac{c_{22}}{\rho}\right)$$
$$c\frac{\partial^2 u}{\partial x\,\partial y}+d\frac{\partial^2 v}{\partial x^2}+b\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 v}{\partial t^2}\qquad\left(d=\frac{c_{66}}{\rho},\quad c=\frac{c_{66}+c_{12}}{\rho}\right)\tag{1.1}$$

where c_{ij} are the elastic constants and ρ is the density of the material.

The solution of Eqs. (1.1) which characterizes elastic waves in an unbounded anisotropic medium as a result of an instantaneous impulse at the origin may be expressed as [1]

$$u(x,y,t)=\sum_{k=1}^{2}R\left\{c\int^{\theta_k}\zeta\lambda_k w_k(\zeta)\,d\zeta\right\}$$
$$v(x,y,t)=\sum_{k=1}^{2}R\left\{\int^{\theta_k}(a\zeta^2+d\lambda_k^{\,2}-1)\,w_k(\zeta)\,d\zeta\right\}\tag{1.2}$$

where the complex variables θ_k are defined by the relations

$$1-\theta_k\xi+\lambda_k\eta=0\qquad(\xi=x/t,\ \eta=y/t)\tag{1.3}$$

The quantities λ_k are determined by the expressions

$$\lambda_k=\left(\frac{[(b+d)-L\theta_k^{\,2}]+(-1)^k\sqrt{Q(\theta_k)}}{2bd}\right)^{1/2}\qquad(k=1,\ 2)\tag{1.4}$$

$$Q(\theta_k)=[(b+d)-L\theta_k^2]^2-4bd\,(1-a\theta_k^2)\,(1-d\theta_k^2),\qquad L=ab+d^2-c^2$$

The functions λ_1 and λ_2 are branches of an algebraic function λ which is single valued on a Riemann surface whose form depends on the relations among the elastic constants. The functions w_1 and w_2 are branches of an analytic function w which is single valued on the Riemann surface. The elastic constants of real media of the indicated class of anisotropy satisfy the inequalities

$$a > d, \quad b > d, \quad d > 0, \quad ab - (c-d)^2 > 0 \tag{1.5}$$

The form of the Riemann surface and the geometric form of the fronts depend on which of the conditions

$$\begin{aligned} N_1 &= (a-d)(b-d) - c^2 > 0, & N_1 &= (a-d)(b-d) - c^2 < 0 \\ N_2 &= (a-d)b - c^2 > 0, & N_2 &= (a-d)b - c^2 < 0 \\ N_3 &= (b-d)a - c^2 > 0, & N_3 &= (b-d)a - c^2 < 0 \end{aligned} \tag{1.6}$$

are satisfied.

The number and location of the branch points of the functions (1.4) in the complex planes have been studied in [3] as functions of the relations among the elastic constants. It should be borne in mind that here

$$\theta_k = \frac{1}{\vartheta_k}, \quad \lambda_k = \theta_k n_k$$

Under the condition $N_2 > 0$ the branch points for the outer radicals of (1.4) are the points $\theta_1 = \pm 1/\sqrt{a}$ for $k = 1$ and the points $\theta_2 = \pm 1/\sqrt{d}$ for $k = 2$.

The branch points for the inner radical of (1.4)

$$\theta_i{}^\circ = \pm \left(\frac{M \pm \sqrt{4bdc^2\,[c^2 - (a-d)(b-d)]}}{K_1 K_2}\right)^{1/2} \tag{1.7}$$

$$K_1 = ab - (c-d)^2, \qquad K_2 = ab - (c+d)^2$$

$$M = (b+d)N_1 - (b-d)(a-b)d$$

may be complex, imaginary, or real. Under the condition $N_1 > 0$ all four occur in complex conjugate pairs; for $N_1 < 0$ all four may be imaginary, or all four real, or two real and two imaginary.

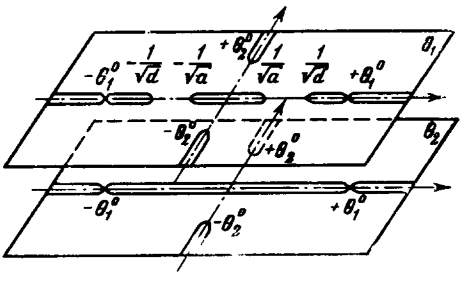

Fig. 1

Under the condition $N_2 < 0$ the branch points for the inner radicals of (1.4) are: for $k = 1$, the points $\theta_1 = \pm 1/\sqrt{a}$ and $\theta_1 = \pm 1/\sqrt{d}$; for $k = 2$ the inner radical has no branch points. Two of the branch points of (1.7) are real, and two are imaginary. The real points of (1.7) for $N_2 > 0$ and for $N_2 < 0$ are in the intervals $(\pm 1/\sqrt{d}, \pm\infty)$.

If the condition $N_2 > 0$ is satisfied, the Riemann surface is constructed in accordance with [1]. For $N_2 < 0$ the function λ_1 is single valued in the complex plane θ_1 with the cuts $(-1/\sqrt{a}, +1/\sqrt{a})$, $(\pm 1/\sqrt{d}, \pm\theta_1{}^\circ)$ and $(\pm\theta_1{}^\circ, \pm\infty)$ along the real axis and $(\pm\theta_2{}^\circ, \pm i\infty)$ along the imaginary axis. The function λ_2 is single valued in the complex plane θ_2 with the cuts $(-\theta_1{}^\circ, +\theta_1{}^\circ)$ and $(\pm\theta_1{}^\circ, \pm\infty)$ along the real axis and $(\pm\theta_2{}^\circ, \pm i\infty)$ along the imaginary axis. By attaching the edges of the cuts $(\pm\theta_1{}^\circ, \pm\infty)$ and $(\pm\theta_2{}^\circ, \pm i\infty)$ of the θ_1 and θ_2 planes in a criss-cross manner we obtain the Riemann surface for single-valued definition of the function λ in the case $N_2 < 0$ (Fig. 1).

Under the condition $N_2 > 0$ the functions λ_1 and λ_2 take on real values on the edges of the cuts of the Riemann surface $(-1/\sqrt{a}, +1/\sqrt{a})$ and $(-1/\sqrt{d}, +1/\sqrt{d})$, whereas if $N_2 < 0$ they are real on $(-1/\sqrt{a}, +1/\sqrt{a})$, $(\pm 1/\sqrt{d}, \pm\theta_1{}^\circ)$ and $(-\theta_1{}^\circ, +\theta_1{}^\circ)$. We specify the values of λ_1 and λ_2 on the Riemann surface by the condition that they are positive for $\theta_k = i\beta$, where β is a sufficiently small positive quantity.

The relations (1.3) establish a correspondence between the points of the xy-plane and the θ_1 and θ_2 planes of the Riemann surface. The wave fronts can be obtained as envelopes of the straight lines (1.3) for real values of θ_k and λ_k

$$x_1 = -\frac{\lambda_1' t}{\lambda_1 - \theta_1\lambda_1'}, \quad y_1 = -\frac{t}{\lambda_1 - \theta_1\lambda_1'} \tag{1.8}$$

$$x_2 = -\frac{\lambda_2' t}{\lambda_2 - \theta_2\lambda_2'}, \quad y_2 = -\frac{t}{\lambda_2 - \theta_2\lambda_2'} \tag{1.9}$$

The normal velocities of propagation of the waves are given by the expressions [3]

$$b_1 = \frac{1}{\sqrt{\theta_1^2 + \lambda_1^2}}, \quad b_2 = \frac{1}{\sqrt{\theta_2^2 + \lambda_2^2}} \tag{1.10}$$

The functions w_k are the branches of an analytic function w which is unique on a two-sheeted Riemann surface, the form of which depends on the relations among the elastic constants. In order that the solution (1.2) correspond to elastic waves in an infinite medium arising from an instantaneous impulsive point load, the function w must be chosen so that the real part of w_1 and w_2 go to zero on the edges of the cuts in the functions λ_1 and λ_2, respectively, when the latter are considered as real functions. Unlike the case $N_2 > 0$, which was considered in [1], these cuts are $(-1/\sqrt{a}, +1/\sqrt{a})$, $(\pm 1/\sqrt{d}, \pm\theta_1^\circ)$ and $(-\theta_1^\circ, +\theta_1^\circ)$ for $N_2 < 0$.

In [1] the properties of the solution corresponding to the condition $N_2 < 0$ were not examined.

2. Geometry of the wave fronts. The conditions for the existence of cuspidal edges. Since the wave fronts are symmetrical with respect to the coordinate axes, it is sufficient to study the portion of the fronts corresponding to the upper edges of the cuts in the θ_1 and θ_2 planes of the Riemann surface which lie on the positive halves of the real axes. Let us select arbitrary points θ_1 and θ_2 on these parts of the edges of the cuts. The points (ξ_1, η_1) and (ξ_2, η_2) on the wave fronts correspond to these points. Denoting by α_1 and α_2 the angles between the negative η-axis and the normals to the fronts at these points, we have

$$\operatorname{tg}\alpha_1 = \frac{\theta_1}{\lambda_1}, \quad \operatorname{tg}\alpha_2 = \frac{\theta_2}{\lambda_2} \tag{2.1}$$

Denoting by β_1 and β_2 the angles between the negative η-axis and the rays connecting the points selected on the wave fronts and the origin, we obtain

$$\operatorname{tg}\beta_1 = -\lambda_1', \qquad \operatorname{tg}\beta_2 = -\lambda_2' \tag{2.2}$$

$$\lambda_k' = \frac{\theta_k\psi_k}{2bd\lambda_k}, \quad \psi_k = (-1)^k \frac{K_1K_2\theta_k^2 - M}{\sqrt{Q(\theta_k)}} - L$$

Under the condition $N_2 > 0$ the points of the wave fronts $\eta_1 = -\sqrt{b}$ and $\eta_2 = -\sqrt{d}$ on the η-axis correspond to the points $\theta_1 = 0$ and $\theta_2 = 0$; the points $\xi_1 = \sqrt{a}$ and $\xi_2 = \sqrt{d}$ on the ξ-axis correspond to $\theta_1 = 1/\sqrt{a}$ and $\theta_2 = 1/\sqrt{d}$. The segments of the fronts whose ends are the above points on the coordinate axes correspond to the values

$$0 \leqslant \theta_1 \leqslant \frac{1}{\sqrt{a}}, \quad 0 \leqslant \theta_2 \leqslant \frac{1}{\sqrt{d}} \tag{2.3}$$

The geometric form of these segments can be investigated with the aid of Eqs. (2.1) and (2.2).

The derivatives of the right sides of (2.1) and (2.2) have the form

$$\left(\frac{\theta_k}{\lambda_k}\right)' = \frac{\varphi_k}{2bd\lambda_k} \tag{2.4}$$

$$\varphi_k = (b+d) + (-1)^k \frac{(b-d)^2 - M\theta_k^2}{\sqrt{Q(\theta_k)}}$$

$$\lambda_k'' = \frac{D_k}{4b^2d^2\lambda_k^3}, \qquad D_k = 2bd(\psi_k + \theta_k\psi_k')\lambda_k^2 - \theta_k^2\psi_k^2$$

In turn, the functions φ_k and D_k have derivatives as follows:

$$\varphi_k' = -\frac{(-1)^k 16bdc^2N_1\theta_k^3}{\sqrt{Q^3}} \tag{2.5}$$

$$D_k' = (-1)^k 48b^2d^2c^2N_1\theta_k\lambda_k^2 \frac{(b-d)^2 - K_1K_2\theta_k^4}{\sqrt{Q^5}}$$

Let us examine the cases which are possible under the condition $N_2 > 0$ according to (1.6).

Case 1. The elastic constants satisfy the condition $N_1 > 0$. For $N_2 > 0$ the function Q is greater than zero on the segments (2.3) as on the segments for the real definition ot the functions (1.4). In accordance with the condition $N_1 > 0$ (1.6) and the expression (2.5), the function φ_1 increases monotonically and φ_2 decreases monotonically. Under the condition $N_2 > 0$, the relations

$$\varphi_1(0) = 2d > 0, \qquad \varphi_2(1/\sqrt{d}) = 2bd/N_2 > 0$$

are satisfied.

Therefore the functions (2.4) have positive values on the intervals (2.3) and the right sides of (2.1) increase monotonically from zero to infinity. The angles α_1 and α_2 therefore increase from 0 to 90° on the intervals (2.3).

Under the condition $N_2 > 0$, the functions D_k have the form

$$D_1(0) = -\frac{4d^2[(b-d)d + c^2]}{b-d}, \qquad D_1\left(\frac{1}{\sqrt{a}}\right) = -\frac{1}{a}\left[\psi_1\left(\frac{1}{\sqrt{a}}\right)\right]^2$$

$$D_2(0) = -\frac{4b^2[(b-d)a - c^2]}{b-d}, \qquad D_2 = \left(\frac{1}{\sqrt{d}}\right) = -\frac{1}{d}\left[\psi_2\left(\frac{1}{\sqrt{d}}\right)\right]^2 \tag{2.6}$$

at the ends of the intervals (2.3).

Since according to (1.6), when $N_1 > 0$, the condition $N_3 > 0$ is satisfied, the functions D_k have negative values at the ends of the intervals (2.3).

The derivative of the function D_1, Eq. (2.5), on the first interval of (2.3) is either smaller than zero or changes sign from negative to positive at the point

$$\theta_0 = \sqrt{b-d}/(K_1K_2)^{1/4} \tag{2.7}$$

Therefore, in both cases the function has negative values on this interval. Then $\lambda_1'' < 0$ and the right side of the first of Eqs. (2.2) increases monotonically from zero to infinity, or the angle β_1 increases monotonically on this interval from 0 to 90°.

The derivative of the function D_2 on the second interval of (2.3) is either greater than zero or changes sign from positive to negative at the point (2.7). In the first case, the function D_2 is a negative, monotonically increasing function. In the second case, the point (2.7) is either imaginary or real. Such a real point will be at the end of the interval (2.3) under study and the function D_2 will be complex there. In the second case, the point (2.7) lies on the interval in question and the function D_2 has a minimum there. If $D_2(\theta_0) < 0$ the function D_2 has negative values on the interval. In these cases $\lambda_2'' < 0$ and the right side of the second of Eqs. (2.2) increases monotonically from zero to infinity on the second interval of (2.3), or the angle β_2 increases monotonically from 0 to 90°.

If the value of the function D_2 is greater than zero at the point (2.7), i.e.

$$[3(b-d)\sqrt{K_1K_2} - 2(b+d)L + M]\sqrt{(b-d)\sqrt{K_1K_2} - M} +$$
$$+ 2[(b+d)\sqrt{K_1K_2} - (b-d)L]\sqrt{2(b-d)\sqrt{K_1K_2}} > 0 \qquad (2.8)$$

then the functions D_2 and λ_2'' change sign twice on the second interval of (2.3), at the points $\theta_2 = \varkappa_1$ and $\theta_2 = \varkappa_2$, where $0 < \varkappa_1 < \theta_0 < \varkappa_2 < 1/\sqrt{d}$. In the intervals $(0, \varkappa_1)$ and $(\varkappa_2, 1/\sqrt{d})$ they are negative, while in the interval $(\varkappa_1, \varkappa_2)$ they are positive. As in the preceding case, the right side of (2.2) assumes values from zero to infinity in the interval in question, but monotonicity of the variation is no longer present; there is a finite maximum at the point $\varkappa_1$, and a finite minimum at $\varkappa_2$. As a result of this, the angle β_2 increases monotonically from zero to the value $\beta_2(\varkappa_1)$ in the interval $(0, \varkappa_1)$, then decreases monotonically from $\beta_2(\varkappa_1)$ to $\beta_2(\varkappa_2)$ in the interval $(\varkappa_1, \varkappa_2)$, and increases monotonically from $\beta_2(\varkappa_2)$ to 90° in the interval $(\varkappa_2, 1/\sqrt{d})$.

If under the condition $N_1 > 0$, the condition (2.8) is not satisfied, the angles α_1, β_1 and α_2, β_2 increase monotonically from 0 to 90° on the intervals (2.3). The portions of the wave fronts in the fourth quadrants of the $\xi\eta$-plane correspond to the intervals (2.3) of the upper edges of the cuts of the Riemann surface. These portions of the wave fronts are convex curves the ends of which approach the coordinate axes at right angles. The wave fronts are closed convex curves with center at the origin. The outer front, corresponding to $k = 1$, is the front of a quasi-longitudinal wave; the inner front, corresponding to $k = 2$, is the front of a quasi-transverse wave.

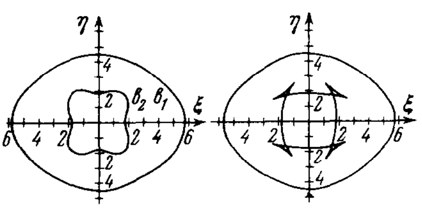

Fig. 2

If the condition (2.8) is satisfied, we arrive at the same result for the quasi-longitudinal wave; the wave front is a closed convex curve. However, we have an entirely different behavior for the quasi-transverse wave. On the interval $(0, 1/\sqrt{d})$ the angle α_2 increases monotonically from 0 to 90°, and the angle β_2 varies continuously. In the interval $(0, \varkappa_1)$ it increases monotonically form 0 to $\beta_2(\varkappa_1)$ decreases monotonically from $\beta_2(\varkappa_1)$ to $\beta_2(\varkappa_2)$ in the interval $(\varkappa_1, \varkappa_2)$ and monotonically increases from $\beta_2(\varkappa_2)$ to 90° in the interval $(\varkappa_2, 1/\sqrt{d})$.

Therefore, the portions of the wave front corresponding to the intervals $(0, \varkappa_1)$ and $(\varkappa_2, 1/\sqrt{d})$ are intersecting convex curves in the fourth quadrant. They proceed from the points $(0, -\sqrt{d})$ and $(\sqrt{d}, 0)$ at right angles to the coordinate axes η and ξ. The portion of the wave front corresponding to the interval $(\varkappa_1, \varkappa_2)$ is a concave curve the ends of which join the ends of the other two portions in cusps of the first kind. Thus, under the condition (2.8) the front of the quasi-transverse wave consists of piecewise smooth curves which form cuspidal edges. The curves of the normal velocities (in km/ /sec) and the wave fronts are shown in Fig. 2 for potassium pentaborate [4]

$c_{11} = 58.2, \quad c_{22} = 35.9, \quad c_{66} = 5.7, \quad c_{12} = 22.9 \quad [10^{10}\ \text{dyne/cm}^2]\ \rho = 1.8\,\text{g/cm}^3$

Case 2. The elastic constants satisfy the condition $N_1 < 0$. Under the condition $N_1 < 0$ the function φ_1 decreases monotonically and the function φ_2 increases monotonically in accordance with (2.5). Since

$$\varphi_1\left(\frac{1}{\sqrt{a}}\right) = \frac{2bd\,(a-d)}{(a-d)\,d + c^2} > 0, \quad \varphi_2(0) = 2b > 0$$

the functions (2.4) are greater than zero on the corresponding intervals (2.3), and the right sides of (2.1) increase monotonically from zero to infinity, or α_1 and α_2 increase monotonically from 0 to 90°.

For $N_1 < 0$ the function D_2' on the second interval of (2.3) is either smaller than zero or changes sign from negative to positive at the point (2.7). As a result of this, the function D_2 either decreases monotonically or has a minimum at the point (2.7).

If the condition $N_3 > 0$ is satisfied, the values of the function D_2 at the ends of the interval are less than zero. In this case, the functions D_2 and λ_2'' are negative on the interval and the angle β_2 increases monotonically from 0 to 90°.

Therefore, under the condition $N_3 > 0$, the front of the quasi-transverse wave is a closed convex curve with center at the origin.

If the condition $N_3 < 0$ is satisfied, the function D_2 has opposite signs at the ends of the interval in question, (2.6), i.e. the functions D_2 and λ_2'' change sign once on the interval, from positive to negative at the point $\theta_2 = \varkappa_1$ (for a real value of (2.7), we have $\varkappa_1 < \theta_0$). The right-hand side of the second of Eqs. (2.2) decreases monotonically from zero to a finite value in the interval $(0, \varkappa_1)$, and increases monotonically from this value to infinity in the interval $(\varkappa_1, 1/\sqrt{d})$, changing sign from negative to positive at the point θ_2^*, which is a zero of the function ψ_2.

Therefore, the angle β_2 decreases monotonically from 0 to $-\beta_2(\varkappa_1)$ in the interval $(0, \varkappa_1)$ and increases monotonically from $-\beta_2(\varkappa_1)$ to 90° in the interval $(\varkappa_1, 1/\sqrt{d})$. It is zero at the point θ_2^*. Since the angle α_2 increases monotonically from 0 to 90° on the interval $(0, 1/\sqrt{d})$ of the upper edge of the cut in the θ_2 plane, a portion of the wave front consisting of two smooth curves joining in a cusp of the first kind corresponds to the interval.

The portion of the front corresponding to the interval $(0, \varkappa_1)$ is a concave curve which proceeds from the point $(0, -\sqrt{d})$ at a right angle to the η-axis and is situated in the third quadrant. The value $\theta_2 = \varkappa_1$ correponds to a cusp in the third quadrant of the $\xi\eta$-plane. The segment corresponding to the interval $(\varkappa_1, 1/\sqrt{d})$ is a convex curve which proceeds from the cusp, cuts the lower η semiaxis above the point $(0, -\sqrt{d})$, and approaches the point $(\sqrt{d}, 0)$ through the fourth quadrant at right angles to the ξ-axis.

Thus, under the condition $N_3 < 0$ (1.6), the front of the quasi-transverse wave consists of piecewise smooth curves forming cuspidal edges at the η-axis.

Under the condition $N_1 < 0$, the function D_1' in the first interval of (2.3) is either greater than zero or changes sign from positive to negative, and the function D_1 either increases monotonically or has a maximum at the point (2.7). According to (2.6), the function D_1 can have positive values at some points of the interval if its value at the point (2.7) is greater than zero, i.e.

$$[3(b-d)\sqrt{K_1K_2} - 2(b+d)L + M]\sqrt{(b-d)\sqrt{K_1K_2} - M} -$$

$$- 2[(b+d)\sqrt{K_1K_2} - (b-d)L]\sqrt{2(b-d)\sqrt{K_1K_2}} > 0 \qquad (2.9)$$

In accordance with (2.5), the function D_1' changes sign on the first interval of (2.3) if

$$(b-d)^2a^2 - K_1K_2 < 0 \tag{2.10}$$

Taking the condition (2.10) into account, we obtain the following inequality:

$$(b+d)\sqrt{K_1K_2} - (b-d)L > (b+d)(b-d)a - (b-d)L = (b-d)[(a-d)d + c^2] > 0 \tag{2.11}$$

For $N_1 < 0$ (1.6), the condition (2.8) is not satisfied, since the function D_2 is less than zero at the point (2.7). Therefore, the condition (2.9) also fails to hold if (2.11) is satisfied. Thus, the functions D_1 and λ_1'' are smaller than zero in the interval under consideration and the angle β_1 increases monotonically from 0 to 90°. The front of the quasi-longitudinal wave front is a closed convex curve.

Let us now examine the case in which the condition $N_2<0$ is satisfied. Here the function λ_1 has real values on the edges of the cuts $(-1/\sqrt{a}, +1/\sqrt{a})$ and $(\pm 1/\sqrt{d}, \pm\theta_1^\circ)$ of the θ_1-plane, and the function λ_2 on the edges of the cut $(-\theta_1^\circ, +\theta_1^\circ)$ of the θ_2-plane (Fig. 1).

We shall investigate the portion of the wave front corresponding to the values $0 \leqslant \theta_2 \leqslant \theta_1^\circ$ of the upper edge of the cut in the θ_2-plane. The points of the front $(0, -\sqrt{d})$ and $(1/\theta_1^\circ, 0)$ located on the η and ξ coordinate axes correspond to the points $\theta_2 = 0$ and $\theta_2 = \theta_1^\circ$. Since for $N_2 < 0$ the condition $N_1 < 0$ is satisfied, by repeating the argument for the case in which the conditions $N_2 > 0$ and $N_1 < 0$ are satisfied we see that in the interval in question the right side of the second of Eqs. (2.1) increases monotonically from zero to the finite value $\theta_1^\circ / \lambda_2(\theta_1^\circ)$, and the angle α_2 increases monotonically from zero to the value $\alpha_2(\theta_1^\circ) < 90^\circ$.

Since under condition $N_3 > 0$ (1.6) the values of the function D_2 at the ends of the interval are smaller than zero, i.e. $D_2(0) = -4b^2N_3$ and $D_2(\theta_1^\circ) = -\infty$ the functions D_2 and λ_2'' have negative values on the interval, and the angle β_2 increases monotonically from 0 to 90°.

Therefore, for $N_3 > 0$ (1.6), corresponding to the interval $(0, \theta_1^\circ)$ there is a portion of the wave front in the fourth quadrant which is a convex curve proceeding from the point $(0, -\sqrt{d})$ at a right angle to the η-axis and approaching the ξ-axis at the point $(1/\theta_1^\circ, 0)$ at an acute angle, since $\alpha_2(\theta_1^\circ) < 90^\circ$

Under the condition $N_3 < 0$ the function D_2 changes sign once from positive to negative in the interval $(0, \theta_1^\circ)$ at the point $\theta_2 = \varkappa_1$, which satisfies the condition $\varkappa_1 < \theta_0$ for a real value of (2.7). The angle β_2 varies in the same way as on the interval $(0, 1/\sqrt{d})$ when the conditions $N_1 < 0$ and $N_3 < 0$ are satisfied, so that $\beta_2(\theta_1^\circ) = 90^\circ$

The portion of the wave front, as in the case for which $N_1 < 0$ and $N_3 < 0$ are satisfied, consists of two smooth curves with opposite signs of curvature which join at a cusp of the first kind. Unlike the case of $N_1 < 0$ and $N_3 < 0$, the convex curve corresponding to the interval $(\varkappa_1, \theta_1^\circ)$ approaches the ξ-axis at the point $(1/\theta_1^\circ, 0)$ at an acute angle, since we have $\alpha_2(\theta_1^\circ) < 90^\circ$.

The entire argument which was carried through for the examination of the interval $(0, 1/\sqrt{a})$ in the case where the conditions $N_1 < 0$ and $N_3 < 0$ were satisfied for $N_2 > 0$ remain valid for the condition $N_2 < 0$. The edges of the cut $(-1/\sqrt{a}, +1/\sqrt{a})$ of the θ_1-plane correspond to the front of the quasi-longitudinal wave, which is a convex closed curve intersecting the coordinate axes at the points $(\pm\sqrt{a}, 0)$ and $(0, \pm\sqrt{b})$.

Let us determine what portion of the wave front corresponds to the values $1/\sqrt{d} \leqslant \theta_1 \leqslant \theta_1^\circ$ of the lower edge of the cut in the θ_1-plane. On this edge of the cut, the function λ_1 has positive values. The points of the wave front $(\sqrt{d}, 0)$ and $(1 / \theta_1^\circ, 0)$ located on the positive ξ-axis correspond to the points $\theta_1 = 1/\sqrt{d}$ and $\theta_1 = \theta_1^\circ$. Since under the condition $N_2 < 0$

$$\varphi_1' < 0, \quad \varphi_1\left(\frac{1}{\sqrt{d}}\right) = \frac{2bd(a-d)}{N_2} < 0$$

the values of the first of Eqs. (2.4) for $k = 1$ are less than zero on the interval in question. The right side of the first of Eqs. (2.1) decreases monotonically from infinity to the value $\theta_1^\circ / \lambda_1(\theta_1^\circ) = \theta_1^\circ / \lambda_2(\theta_1^\circ)$ and the angle α_1 decreases monotonically from $90°$ to the value $\alpha_1(\theta_1^\circ) = \alpha_2(\theta_1^\circ)$. For $N_2 < 0$ the function D_1' is either greater than zero or changes sign from positive to negative on the interval in question. However, the latter possibility is excluded, since at the ends of the interval

$$D_1(1/\sqrt{d}) = -[\psi_1(1/\sqrt{d})]^2 / d < 0, \qquad D_1(\theta_1^\circ) = +\infty$$

and the function D_1 cannot have a maximum within the interval. Therefore, the functions D_1 and λ_1'' change sign from negative to positive at a point $\theta_1 = \varkappa$.

In the interval $(1/\sqrt{d}, \varkappa)$, the right side of the first of Eqs. (2.2) increases monotonically from $-\infty$ to the value $[-\lambda_1'(\varkappa)] < 0$, and the angle β_1 increases monotonically from $90°$ to $\beta_1(\varkappa)$.

In the interval $(\varkappa, \theta_1^\circ)$, the right side of the first of Eqs. (2.2) decreases monotonically from the value $-\lambda_1'(\varkappa)$ to $-\infty$ and the angle β_1 decreases monotonically from the value $\beta_1(\varkappa)$ to $90°$. Therefore, the portion of the wave front in the first quadrant corresponds to the lower edge of the cut $(1/\sqrt{d}, \theta_1^\circ)$.

One segment of this portion of the front, the one corresponding to the interval $(\theta_1^\circ, \varkappa)$, is a convex curve which proceeds from the point $(1/\theta_1^\circ, 0)$ at an acute angle to the ξ-axis and is the continuation of a convex portion of the wave front corresponding to the interval $(0, \theta_1^\circ)$ of the upper edge of the cut in the complex plane θ_2, since $\alpha_1(\theta_1^\circ) = \alpha_2(\theta_1^\circ)$.

The other segment, corresponding to the interval $(\varkappa, 1/\sqrt{d})$, is a concave curve, one end of which is joined to the first segment at a cusp of the first kind, and the other end of which approaches the ξ-axis at a right angle at the point $(\sqrt{d}, 0)$.

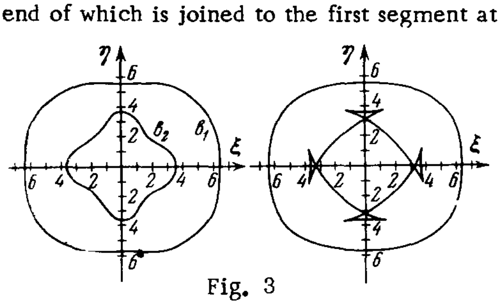

Fig. 3

Thus, if the condition $N_2 < 0$ is satisfied, the front of the quasi-transverse wave consists of piecewise smooth curves which form cuspidal edges and are expressed by Eqs. (1.9) on the edges of the cut $(-\theta_1^\circ, +\theta_1^\circ)$ and by Eqs. (1.8) on the edges of the cuts $(\pm 1/\sqrt{d}, \pm\theta_1^\circ)$ of the θ_1-plane of the Reimann surface. If $N_2 < 0$ and $N_3 > 0$, the wave front has cuspidal edges at the ξ-axis; if $N_2 < 0$ and $N_3 < 0$, there are cuspidal edges at the ξ- and η-axes. Curves of normal velocities in km/sec and the wave fronts are shown in Fig. 3 for a medium satisfying the condition $N_2 < 0$ and $N_3 < 0$.

As an example we take magnesium sulfate heptahydrate [4]

$$C_{11} = 69.8, \quad c_{22} = 52.9, \quad c_{66} = 22.2, \quad c_{12} = 39, \; \rho = 1.7 \text{ g/cm}^3$$

The investigation which has just been completed makes it possible: (i) to establish a correspondence between the points of the wave fronts and points on the Reimann surface; (ii) to study the geometric properties of the wave fronts as they depend on the elastic constants of the medium; and (iii) to establish the conditions for existence of cuspidal edges, $N_2 < 0$, $N_3 < 0$, and (2.8).

The problem of the geometry of the wave fronts has attracted the attention of a number of authors; brief remarks on the basic works are given in [5]. According to [5], the condition for the existence of cuspidal edges for media with $N_1 > 0$ may be expressed as

$$B^2 (B - 2N_1 / d) - 4N_1^2 (B - N_1 / d) / cd < 0 \qquad (2.12)$$

$$B = (a + b) + 2 (c - d)$$

For potassium pentaborate, ice, cobalt, and beryl [5] the conditions (2.8) and (2.12) both assert the presence of cuspidal edges. For beryl with the elastic constants [4]

$$c_{11} = 26.8, \quad c_{33} = 23.5, \quad c_{44} = 6.55, \quad c_{13} = 6.66$$

and for potassium bromide for all values of the elastic constants [4], the conditions (2.8) and (2.12) lead to different results; the condition (2.8) says that there are cuspidal edges, while (2.12) denies that there are any. The form of the wave front for potassium bromide and the results of experiment presented by Aleksandrov in [7] convincingly confirm the presence of cuspidal edges on the wave surface of potassium bromide. The construction of the wave front for potassium bromide according to Eqs. (1.8) and (1.9) leads to the same result. All of this gives grounds for stating that Khatkevich [5] made an error in the derivation of the condition (2.12). We remark that Khatkevich [5] correctly pointed out an erroneous statement in [8] denying the possibility of satisfying inequalities of the type $N_2 < 0$ and $N_3 < 0$. For clarity we add that this error was noted considerably earlier by the author himself [8] and is examined in [3].

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